JOURNAL OF ALGEBRA 136, 401–431 (1991)

Ordinary and p-Modular Character Degrees of Solvable Groups*

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Communicated by Gernot Stroth

Received October 3, 1989

INTRODUCTION

The Fong–Swan theorem [1, X, 2.1] shows a relation between irreducible Brauer characters and ordinary irreducible characters by the following: Let φ be an irreducible Brauer character of a p-solvable group G. There exists a p-rational irreducible character χ of G such that $\chi = \varphi$ as a Brauer character. Especially: Every condition on ordinary characters is valid for Brauer characters (in a p-solvable group). We ask now for a kind of inversion of this relation and consider the character degrees. We denote by

$$cd(G) = \{\chi(1) \mid \chi \in Irr(G)\}$$

the set of degrees of ordinary irreducible characters and by

$$cd_p(G) = \{\beta(1) \mid \beta \in IBr_p(G)\}$$

the set of degrees of irreducible Brauer characters. If the group G is p-solvable, the Fong–Swan theorem yields:

$$cd_p(G) \subseteq cd(G).$$

In this paper we assume some arithmetical conditions on $cd_p(G)$ and we ask for arithmetical conditions on $cd(G)$. Precisely: Assume that all elements of $cd_p(G)$ are squarefree ($q^2 \nmid \beta(1)$ for all $\beta \in IBr_p(G)$ and all primes $q \mid |G|$). It turns out that all elements of $cd(G)$ are cubefree (Sect. 2). We generalize this problem in Section 3 assuming that for all $\beta \in IBr_p(G)$ and all primes $q \mid |G|$ the following holds: $q^n \nmid \beta(1)$. Because this assumption

* This is a part of a doctoral thesis at Mainz. The author thanks the Deutsche Forschungsgemeinschaft for its support.

0021-8693/91 $3.00

is very weak for large $n \in \mathbb{N}$, we get only a weak result for the group structure of G and for $cd(G)$.

I want to thank my teacher Dr. B. Huppert for many ideas and discussions during the time I worked on this dissertation.

0. Propositions

The methods which are used in this paper depend essentially on the solvability of the group G. In many cases we have to consider nilpotent factors of G. Therefore, we use the following definition:

0.1. Definition. Let G be finite and solvable.

(a) We denote by $F(G)$ the Fitting subgroup of G and define normal subgroups $F_j(G) \trianglelefteq G$ by the following: $F_0 = E$ and

$$F_j(G)/F_{j-1}(G) := F(G/F_{j-1}(G)), \qquad j = 1, 2, \ldots.$$

The minimal number $n \in \mathbb{N}$ such that $F_n(G) = G$ is called the nilpotent length of G and is denoted by $n = n(G)$. For all $1 \leqslant j \leqslant n(G)$ the following holds:

$$C_{G/F_{j-1}(G)}(F_j(G)/F_{j-1}(G)) \leqslant F_j(G)/F_{j-1}(G) \qquad [2, \text{III}, 4.2(\text{b})].$$

(b) We denote by $\Phi(G)$ the Frattini subgroup of G and define normal subgroups $\Phi_j(G) \trianglelefteq G$ by

$$\Phi_j(G)/F_{j-1}(G) := \Phi(G/F_{j-1}(G)), \qquad j = 1, 2, \ldots.$$

A theorem of Gaschütz [2, III.4.5] yields: $F_j(G)/\Phi_j(G) = F(G/\Phi_j(G))$ is the direct product of minimal abelian normal subgroups of $G/\Phi_j(G)$.

0.2. *Remark.* It follows for every subnormal subgroup $N \trianglelefteq\trianglelefteq G$ with $F_j(G) \leqslant N$ that $F_j(N) = F_j(G)$ $(j = 0, 1, 2, \ldots)$.

We find an essential key, used in this paper, in the following question: Let $N \trianglelefteq G$ and $\alpha \in IBr_p(N)$. Is it possible to control the inertia subgroup $T_G(\alpha)$? Precisely: Is it possible to control the operation of G/N on the set $IBr_p(N)$? In the case of $p \nmid |N|$, we have $IBr_p(N) = Irr(N)$ (cf. Isaacs [6, 15.13]) and the ordinary representation theory yields many aids to answer this question.

We conclude the propositions considering a theorem of Huppert which is a special case and the starting point of our problem.

0.3. Satz (Huppert). *Let G be p-solvable, $O_p(G) = E$, and $cd_p(G)$*

contains only 1 *and some primes. Then* $cd(G)$ *contains only* 1 *and some primes, except one case*:

$$p=3, \qquad G'\cong SL(2,3), \qquad |G/G'Z(G)|=2.$$

Conversely, if

$$G'\cong SL(2,3) \qquad \textit{and} \qquad |G/G'Z(G)|=2,$$

then

$$cd_3(G)=\{1,2,3\} \qquad \textit{and} \qquad cd(G)=\{1,2,3,4\}.$$

Proof. Huppert [4, Thm. 1].

1. Some Tools

Before we consider the case that all elements of $cd_p(G)$ are squarefree, we heap up some tools which help us to get the proof of our main theorem clear. They are all valid without assumptions on $cd_p(G)$.

1.1. Lemma. *Let* $N \trianglelefteq G$ *and* V *an irreducible* KN*-module for any field* K. *Further let* $T := T_G(V)$ *be the inertia subgroup of* V *and* W *an irreducible* KT*-module such that* $W_N = eV$. *Then* W^G *is an irreducible* KG*-module.*

Proof. Manz [8, Lemma 1].

1.2. Corollary. *Let* $N \trianglelefteq G$ *and* V *an irreducible* KN*-module for any field* K. *There exists an irreducible* KG*-module* X *such that*

$$X_N = V \oplus \ldots \qquad \textit{and} \qquad |G:T_G(V)|\cdot \dim V \mid \dim X.$$

Proof. Lemma 1.1.

The following theorem is a generalization of a theorem which is well known in characteristic 0. We do not give a proof, because it is nearly the same as in characteristic 0.

1.3. Theorem. *Let* $N \trianglelefteq G$, V *an irreducible* KN*-module for an algebraic closed field* K, *and* $T_G(V)=G$. *Assume that for all* $q \mid |G/N|$ (q *a prime*) *and* $Q/N \in Syl_q(G/N)$ *there exists an irreducible* KQ*-module* W *such that* $W_N = V$. *Then there exists an irreducible* KG*-module* X *with* $X_N = V$.

1.4. Corollary. *Let* $N \trianglelefteq G$ *and* V *an irreducible* KN*-module for an algebraic closed field* K. *Assume that* $T_G(V)=G$ *and that all Sylow* *subgroups of G/N are cyclic. Then there exists an irreducible KG-module X with $X_N = V$.*

Proof. Huppert and Blackburn [3, VII, 9.9a)] and Theorem 1.3.

1.5. LEMMA. *Let A be an abelian normal subgroup of G. We denote with $\hat{A} := Irr(A)$ the character group of A. $\bar{G} := G/A$ acts on $\hat{A}$ by $\lambda^{\bar{g}}(a) = \lambda(a^{\bar{g}^{-1}})$ for any $\lambda \in \hat{A}$, $a \in A$, and $\bar{g} \in \bar{G}$.*

(a) *If A is elementary abelian, then A is an irreducible $\bar{G}$-module if and only if $\hat{A}$ is an irreducible $\bar{G}$-module.*

(b) $C_{\bar{G}}(A) = C_{\bar{G}}(\hat{A})$.

(c) *Let $\bar{g} \in \bar{G}$. The number of fixpoints of $\bar{g}$ on A is the same as on $\hat{A}$. Especially: $\bar{G}$ acts fixpointfreely on A if and only if $\bar{G}$ acts fixpointfreely on $\hat{A}$.*

1.6. LEMMA. *Let A, V be abelian groups with $(|A|, |V|) = 1$ and A acts faithfully on V. Then A has a regular orbit on V.*

Proof. Passman [9, 2.2].

1.7. LEMMA. *Let N be an abelian normal subgroup of G with $(|G/C_G(N)|, |N|) = 1$ and $G/C_G(N)$ abelian.*

(a) *$G/C_G(N)$ has a regular orbit on N, $Irr(N)$, and $IBr_p(N)$ if $p \nmid |N|$.*

(b) *There exists a $\chi \in Irr(G)$ resp. $\beta \in IBr_p(G)$ if $p \nmid |N|$ such that*

$$|G/C_G(N)| \mid \chi(1) \text{ resp. } |G/C_G(N)| \mid \beta(1).$$

Proof. (a) $\mathrm{Irr}(N)$ and N are permutation isomorphic [6, 13.24]. Lemma 1.6 yields our assertion.

(b) It follows by (a) and Corollary 1.2.

1.8. LEMMA. *Let $P \trianglelefteq G$ and $|P| = p$ (p a prime). Further let V be an irreducible G-module over a finite field and $C_P(V) = E$. Then*

(a) $p \nmid |V|$;

(b) *P acts fixpointfreely on V.*

Proof. (a) Clear, because $O_p(G) \leqslant \ker V$ if char $V = p$.

(b) Assume that $vg = v$ for any $0 \neq v \in V$ and $1 \neq g \in P$. Since $P = \langle g \rangle$ it follows that $\langle v \rangle$ is a trivial P-submodule of V. Clifford theory yields now that P acts trivial on V [2, V, 17.3(f)], a contradiction.

1.9. LEMMA. *Let V be a semisimple KG-module (for any field K).*

Further let $N \lhd G$ *and assume that for* all $0 \neq v \in V$ *the following three conditions are valid*:

(i) $G = C_G(v)N$;

(ii) $C_G(v) \cap N = E$;

(iii) $C_G(v)$ *is not normal in* G.

Then V *is an irreducible* KG*-module and is irreducible as a* KN*-module.*

Proof. The assumptions (i) and (ii) yield for all $0 \neq v \in V$

$$|G/N| = |C_G(v)N/N| = |C_G(v)/(C_G(v) \cap N)| = |C_G(v)|.$$

Assume that V is not irreducible. This means $V = W_1 \oplus W_2 \oplus \cdots$ with irreducible KG-modules W_j. Let $0 \neq w_1 \in W_1$ and assume further that $C_G(w_1) = C_G(w_2)$ for all $0 \neq w_2 \in W_2$. Hence $C_G(w_1) \leqslant C_G(W_2) \leqslant C_G(w_2)$ and therefore $C_G(w_1) = C_G(W_2)$ is normal in G, a contradiction to (iii). Hence there exists a $0 \neq w_2 \in W_2$ such that $C_G(w_1) \neq C_G(w_2)$ and we get

$$\begin{aligned} |G/N| = |C_G(w_1 + w_2)| &= |C_G(w_1) \cap C_G(w_2)| \\ &< |C_G(w_1)| = |G/N|, \qquad \text{contradiction.} \end{aligned}$$

Therefore V is an irreducible KG-module and for all $0 \neq v \in V$ it holds that

$$\begin{aligned} V &= \langle vgh \mid g \in C_G(v), h \in N \rangle \\ &= \langle vh \mid h \in N \rangle. \end{aligned}$$

So, V is an irreducible KN-module.

1.10. Lemma. *Let* $G/F(G)$ *be a* p*-group* (*for a prime* p). *We define* P, $P^* \trianglelefteq G$ *by*

$$F(G) = P \times P^* \qquad \textit{with} \quad |P| = p^{\cdot\cdot} \textit{ and } p \nmid |P^*|.$$

Then $C_G(P^*) \leqslant F(G)$.

Proof. Since $P^* C_G(P^*) = P^* \times P_1$ where $P_1 \trianglelefteq G$ is a normal p-subgroup of G ($C_G(P^*) \trianglelefteq G$), it follows that

$$C_G(P^*) \leqslant P^* \times P_1 \leqslant F(G).$$

1.11. Lemma. *Let* G *be solvable and* $M \trianglelefteq G$ *such that* $F(G) \leqslant M$ *and* $M/F(G)$ *is a cyclic* p*-group. Then there exists a normal subgroup* $N \trianglelefteq G$ *such that* $F(G)/N$ *is a chief factor in* G *and*

$$C_M(F(G)/N) \leqslant F(G).$$

Proof. Without loss of generality $\Phi(G)=E$. Hence $F(G)=N_1\times\cdots\times N_k$ is the direct product of minimal normal subgroups $N_j \trianglelefteq G$. Since $F(G)=F(M)$, we have

$$F(G)=C_M(F(G))=\bigcap_{i=1}^{k} C_M(N_j) \qquad [2, \text{III.4.2(b)}].$$

Hence $C_M(N_j)=F(G)$ for at least one $N_j \trianglelefteq G$ because $M/F(G)$ is a cyclic p-group.

1.12. LEMMA. *Let G be p-solvable with $|O_{p'p}(G)/O_{p'}(G)|=p^n$. Then*

$$G/O_{p'p}(G) \lesssim GL(n,p).$$

Proof. It is an easy consequence of the lemma of Hall and Higman [2, VI, 6.5].

We conclude this section with a remark about the set of character degrees of p-nilpotent groups.

1.13. LEMMA. *Let G be solvable with an abelian Sylow p-subgroup. Then*

$$cd(O_{p'p}(G))=cd_p(O_{p'p}(G)).$$

Proof. Feit [1, X, 1.7].

2. SQUAREFREE p-MODULAR CHARACTER DEGREES

In this section we consider groups with squarefree p-modular character degrees. It turns out that the ordinary character degrees are cubefree. To prove this we need a sound knowledge of the group structure.

2.1. DEFINITION. Let $n=\prod_{i=1}^{k} p_i^{a_i}$ be the prime factor decomposition of $n\in\mathbb{N}$. We define

(a) $\tau(n)=\max\{a_i \mid i=1,\ldots,k\}$;

(b) $\tau(G)=\max\{\tau(\chi(1)) \mid \chi\in Irr(G)\}$;

(c) $\tau_p(G)=\max\{\tau(\beta(1)) \mid \beta\in IBr_p(G)\}$.

2.2. *Remark.* (a) $\tau(G)\leqslant s$ resp. $\tau_p(G)\leqslant s$ (for $s\in\mathbb{N}$) is inheritable to subnormal subgroups (Clifford) and factor groups.

(b) G has squarefree (p-modular) character degrees, if and only if $\tau(G)\leqslant 1$ (resp. $\tau_p(G)\leqslant 1$).

Huppert and Manz considered the case that all ordinary character degrees are squarefree [5]. They showed that a solvable group G has the following structure:

2.3. THEOREM. *Let G be solvable and $\tau(G)=1$.*

(a) *$F_1(G)=F(G)$ is metabelian.*

(b) *$|F_{j+1}(G)|F_j(G)|$ is squarefree; in particular $F_{j+1}(G)/F_j(G)$ is cyclic.*

(c) *$F_3(G)=G$. Hence $n(G)\leqslant 3$ and $dl(G)\leqslant 4$.*

Proof. Huppert and Manz [5, Thm. 1.3].

2.4. THEOREM. *Let G be solvable and $O_p(G)=E$. Suppose further $\tau_p(G)=1$. We put $F_j:=F_j(G)$. The following assertions hold:*

(a) *F_1 is metabelian.*

(b) *F_2/F_1 is cyclic of squarefree order.*

(c) *G/F_2 is abelian; hence $n(G)\leqslant 3$ and $dl(G)\leqslant 4$.*

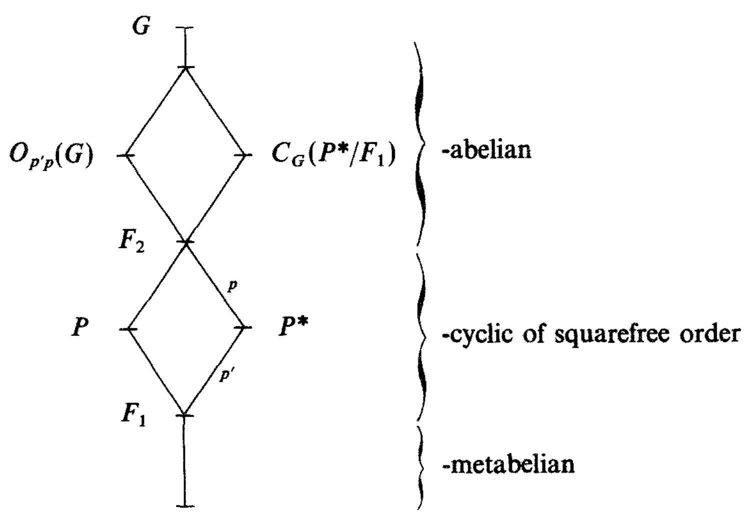
(d) *$p^2 \nmid |G/F_2|$; in particular $p^3 \nmid |G|$.*

(e) *$F_2 \leqslant O_{p'p}(G)$ and $O_{p'p}(G)/F_2$ is cyclic of squarefree order.*

(f) *$G/O_{p'p}(G)$ is cyclic and $|G/O_{p'p}(G)| \mid p-1$.*

(g) *Let $P/F_1 \in Syl_p(F_2/F_1)$ and $F_2/F_1 = P/F_1 \times P^*/F_1$. ($P^*/F_1$ is the p'-part of F_2/F_1.) Then $|G/C_G(P^*/F_1)|$ is squarefree.*

(h) *Let $P, P^* \trianglelefteq G$ be defined as in (g). Then $O_{p'p}(G)\cap C_G(P^*/F_1)=F_2$.*



Proof. (a) $\tau(F_1)=\tau_p(F_1)\leqslant 1$, because $p\nmid F_1|$ and hence Theorem 2.3(a) yields our statement.

For the rest of this proof we put $\Phi(G)=E$ (w.l.o.g.), because of $F(G/\Phi(G))=F(G)/\Phi(G)$ and $O_{p'p}(G/\Phi(G))=O_{p'p}(G)/\Phi(G)$ $(p\nmid|F_1|)$. We assume further $p\mid|G|$, otherwise $\tau(G)=1$ and the assertions follow by Theorem 2.3. In particular $G>F_1$.

(b) Let $Q/F_1\in Syl_q(F_2/F_1)$ for any prime $q\mid|F_2/F_1|$. We define a subnormal subgroup $A\trianglelefteq\trianglelefteq G$ such that A/F_1 is a maximal normal subgroup of Q/F_1. Then $F(A)=F_1$ and $\tau_p(A)=1$. Further, A/F_1 acts faithfully on the q'-part of F_1 (Lemma 1.10). Since F_1 is abelian ($\Phi(G)=E$), there exists a $\beta\in IBr_p(A)$ with $|A/F_1|\mid\beta(1)$ (Lemma 1.7(b)); hence $|A/F_1|=q$ and $Q=A$ because A/F_1 was maximally chosen. Therefore $|F_2/F_1|$ is squarefree.

Part (b) yields that F_2 is p-nilpotent $(p\nmid|F_1|)$ with abelian Sylow p-subgroups and therefore $\tau(F_2)=\tau_p(F_2)=1$ (Lemma 1.13). In the following we assume w.l.o.g. $G>F_2$.

(c) Let $Q_i/F_1\in Syl_{q_i}(F_2/F_1)$, where q_i are different primes dividing $|F_2/F_1|$. Since $|F_2/F_1|$ is squarefree we get

$$\begin{aligned} G/F_2 &= G\bigg/\left(\bigcap_{i=1}^{k} C_G(Q_i/F_1)\right)\\ &\lesssim G/C_G(Q_1/F_1)\times\cdots\times G/C_G(Q_k/F_1)\\ &\lesssim GF(q_1)^{\times}\times\cdots\times GF(q_k)^{\times}\end{aligned}$$

and G/F_2 is abelian.

(d) Assume $p\mid|G/F_2|$ and let $H/F_2\in Syl_p(G/F_2)$. H/F_2 acts faithfully on the p'-part of F_2/F_1 (Lemma 1.10) and Lemma 1.7(b) yields a $\beta\in IBr_p(H/F_1)$ such that $|H/F_2|\mid\beta(1)$. Since $\tau_p(H)=1$, $|H/F_2|=p$ holds. By (b) $|G|_p\mid p^2$ follows.

(e) F_2 is p-nilpotent, hence $F_2\leqslant O_{p'p}(G)$. Since $p^3\nmid|G|$, the Sylow p-subgroups of G are abelian and Lemma 1.13 yields

$$\tau(O_{p'p}(G))=\tau_p(O_{p'p}(G))=1.$$

Therefore $O_{p'p}(G)/F_2$ is cyclic of squarefree order by Theorem 2.3(b).

(f) Let K/F_2 be the Hall p'-subgroup of G/F_2. Then

$$G=O_{p'p}(G)\cdot K$$

and therefore

$$G/O_{p'p}(G)=KO_{p'p}(G)/O_{p'p}(G)\cong K/(O_{p'p}(G)\cap K)=K/O_{p'p}(K).$$

By construction, $p^2 \nmid |K|$ and Lemma 1.12 yields

$$K/O_{p'p}(K) \lesssim GF(p)^{\times} \qquad \text{cyclic.}$$

(g) We assume (w.l.o.g.) that G/F_2 is an r-group for a prime $r \neq p$. For all $U \leqslant G$ we put $\bar{U} := UF_1/F_1$. Further let $\bar{R} \in Syl_r(\overline{F_2})$ and $\bar{R}^* \trianglelefteq \bar{G}$ with $\overline{F_2} = \bar{R} \times \bar{R}^*$. For $\bar{P} \in Syl_p(\overline{F_2})$ we define $\bar{P}^*$ analogous. Since $\overline{F_2}$ is abelian, G/F_2 is an r-group and $|\bar{R}| \mid r$, $\bar{R} \leqslant Z(\bar{G})$ holds and hence

$$C_{\bar{G}}(\bar{R}^* \cap \bar{P}^*) = C_{\bar{G}}(\bar{P}^*).$$

Lemma 1.7(b) yields a $\beta \in IBr_p(\bar{G})$ such that $|\bar{G}/(C_{\bar{G}}(\bar{R}^* \cap \bar{P}^*))| \mid \beta(1)$ and statement (g) is proved.

(h) We consider again $\bar{G} := G/F_1$. Since $|\bar{P}| \mid p$ and $\overline{O_{p'p}(G)}$ is p-nilpotent, $\overline{O_{p'p}(G)} \leqslant C_{\bar{G}}(\bar{P})$ follows. Therefore

$$\overline{O_{p'p}(G)} \cap C_{\bar{G}}(\bar{P}^*) = \overline{F_2}$$

holds, because $C_{\bar{G}}(\overline{F_2}) = \overline{F_2}$ and $\overline{F_2} \leqslant \overline{O_{p'p}(G)}$.

2.5. *Remark.* Let G be a solvable group with $O_p(G) = E$ and squarefree p-modular character degrees. Theorems 2.3 and 2.4 show that the structure of G is only a little different from such solvable groups, whose ordinary character degrees are squarefree. Solely the last Fitting factor shows different. If H is a solvable group with $\tau(H) = 1$, then $H/F_2(H)$ is cyclic of squarefree order. $G/F_2(G)$ is abelian, but not necessarily cyclic. But it turns out that $|G/F_2(G)|$ is cubefree.

2.6. MAIN THEOREM. *Let G be solvable, $O_p(G) = E$, and $\tau_p(G) \leqslant 1$. Then $\tau(G) \leqslant 2$.*

2.7. COROLLARY. *Let G be solvable, $O_p(G) = E$, and $\tau_p(G) = 1$. Then $|G/F_2(G)|$ is cubefree.*

Proof. By Theorem 2.4(c) $G/F_2(G)$ is abelian. Lemma 1.1 of Huppert and Manz [5] yields now a $\chi \in \operatorname{Irr}(G/F_1(G))$ such that $|G/F_2(G)| \mid \chi(1)$. Our main theorem yields that $|G/F_2(G)|$ is cubefree.

By proving our main theorem we consider minimal counterexamples. It turns out that these groups very often have a special semilinear structure. The following theorem is a central key to may of our proofs.

2.8. THEOREM. *Assume that G has a normal Sylow p-subgroup P such that $|P| = p$ and $C_G(P) = P$. We put $K := GF(q)$ (q a prime). Let V be an* *irreducible faithful KG-module such that* $\dim V = n$. *Assume further that for* all $v \in V$ $G = C_G(v) \cdot P$ *holds. Then*

(a) $|G/P|$ *is a prime.*

(b) $p = (q^n - 1)/(q^{n/r} - 1)$ *with* $r = |G/P|$.

Proof. (a), (b): For all $v \in V$ the following holds:

$$\begin{aligned} V &= \langle vgh \mid g \in C_G(v), h \in P \rangle \\ &= \langle vh \mid h \in P \rangle. \end{aligned}$$

Therefore V is an irreducible KP-module. By Huppert [2, II, 3.11] we can identify V with some field $GF(q^n)^+$ and P acts by multiplication

$$v \to av \qquad (a, v \in GF(q^n), a^p = 1 \neq a).$$

Further

$$G/C_G(P) = G/P \lesssim Aut(GF(q^n): GF(q))$$

and $r := |G/P| \mid n$. Since n is the minimal number such that $p \mid q^n - 1$ [2, II, 3.10] we get

$$p \,\Big|\, \frac{q^n - 1}{q^{n/r} - 1}.$$

Every complement $R \cong G/P$ is cyclic and generated by a uniquely determined element ρ such that

$$v\rho = b_\rho v^{q^{n/r}} \qquad (v, b_\rho \in GF(q^n), b_\rho \neq 0).$$

Hence we obtain $|\{b_\rho\}| = p$. By our assumption, there exists for every $0 \neq v \in V$ a complement $\langle \rho \rangle$ such that

$$v = v\rho = b_\rho v^{q^{n/r}}$$

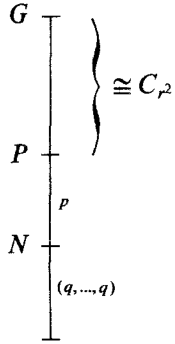
and therefore $\{v^{1 - q^{n/r}}\} \subseteq \{b_\rho\}$. In particular

$$p \,\Big|\, \frac{q^n - 1}{q^{n/r} - 1} = |\{v^{1 - q^{n/r}}\}| \leqslant p$$

and we finally obtain $p = (q^n - 1)/(q^{n/r} - 1)$. This obviously forces r to be a prime.

2.9. COROLLARY. *Let N be a minimal normal subgroup of G and* $P \trianglelefteq G$

a normal subgroup such that $|P/N| = p$. *Further let* $G/P \cong C_{r^2}$ *cyclic of order* r^2, $C_G(P/N) = P$, *and* $C_P(N) = N$.



Then G/N *acts faithfully on* N *and there exists a* $\beta \in IBr_p(G)$ *with* $r \mid \beta(1)$.

Proof. Obviously G/N is a Frobenius group with kernel P/N and complement isomorphic to G/P. Since $C_{P/N}(N) = E$ (and $C_{G/N}(N) \trianglelefteq G/N$), G/N acts faithfully on N.

Asume $r \nmid \beta(1)$ for every $\beta \in IBr_p(G)$. By Corollary 1.2 we obtain for all $\lambda \in IBr_p(N)$:

$$r \nmid |G : T_G(\lambda)|, \qquad \text{hence} \quad G/N = T_{G/N}(\lambda) P/N.$$

Since G/N acts faithfully on N we have $p \nmid |N|$ and therefore $IBr_p(N)$ is an irreducible faithful G/N-module (Lemma 1.5(a), (b)). By Theorem 2.8, this implies $|(G/N)/(P/N)| = |G/P|$ is a prime; contradiction.

2.10. LEMMA. *Let* G *be solvable,* $O_p(G) = E$, *and* $\tau_p(G) = 1$. *We put* $F_j = F_j(G)$. *Suppose the following*:

(1) $F_2 = O_{p'p}(G)$;

(2) $|G/F_2| = r^s$, r *is a prime and* $2 \leqslant s$;

(3) $r \nmid |F_2/F_1|$.

Let $P/F_1 \in Syl_p(F_2/F_1)$. *By* (1) *and* (2) *and Theorem* 2.4(b) *we obtain* $|P/F_1| = p$. *Further* $G/O_{p'p}(G) = G/F_2$ *is cyclic* (*Theorem* 2.4(f)) *and there exists a uniquely determined normal subgroup* K *with* $|K/F_2| = r^{s-1}$.

(a) *There exists a normal subgroup* $L \trianglelefteq G$ *with*

$$K/P = L/P \times F_2/P \qquad \textit{and} \qquad C_{L/F_1}(P/F_1) = P/F_1.$$

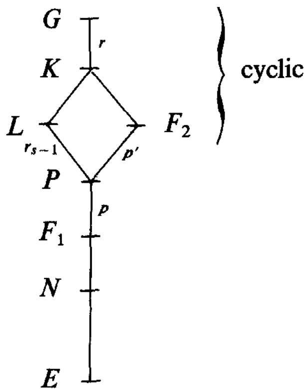
$L/P \cong K/F_2$ *is cyclic of order* r^{s-1}.

(b) *There exists a normal subgroup* $N \trianglelefteq G$ *with* $N < F_1$ *such that* F_1/N *is a chief factor in* P *and* $C_P(F_1/N) = F_1$. *Further, for every* $1 \neq \lambda \in IBr_p(F_1/N)$ *the following hold*:

(i) $L/F_1 = T_{L/F_1}(\lambda)P/F_1$;

(ii) $P/F_1 \cap T_{L/F_1}(\lambda) = E$;

(iii) $T_{L/F_1}(\lambda)$ *is not normal in* L/F_1.



Proof. We put $\bar{U} := UF_1/F_1$ for all $U \leqslant G$.

(a) Let $P^* \trianglelefteq G$ again be defined by $\bar{P} \times \bar{P}^* = \overline{F_2}$. By Theorem 2.4(g), $|\bar{G}/C_{\bar{G}}(\overline{P^*})$ is squarefree and hence

$$\bar{K} \leqslant C_{\bar{G}}(\overline{P^*}).$$

As $(|\bar{K}/\overline{F_2}|, |\overline{F_2}|) = 1$, there exists a normal subgroup $L \trianglelefteq K$ such that

$$\bar{K}/\bar{P} = \bar{L}/\bar{P} \times \overline{F_2}/\bar{P} \qquad \text{(Zassenhaus).}$$

Since $\bar{L}$ is a normal Hall $\{r, p\}$-subgroup of $\bar{K}$, we obtain $\bar{L} \trianglelefteq \bar{G}$ and

$$O_{p'p}(L) = O_{p'p}(G) \cap L = F_2 \cap L = P.$$

Therefore $C_{\bar{L}}(\bar{P}) = \bar{P}$.

(b) By Lemma 1.11, there exists a normal subgroup $N \trianglelefteq G$ such that F_1/N is a chief factor in G and $C_{\bar{P}}(F_1/N) = \bar{E}$.

(i) The assumption $\tau_p(G) = 1$ implies for all $\lambda \in IBr_p(F_1/N)$ that $r^2 \nmid |\bar{G} : T_{\bar{G}}(\lambda)|$ and in particular

$$\bar{K} \leqslant T_{\bar{G}}(\lambda)\overline{F_2} \qquad \text{for all} \quad \lambda \in IBr_p(F_1/N).$$

Therefore

$$\bar{L} = T_{\bar{L}}(\lambda)\bar{P} \qquad \text{for all} \quad \lambda \in IBr_p(F_1/N).$$

(ii) By Lemma 1.8(b), $\bar{P}$ acts faithfully on F_1/N and on $IBr_p(F_1/N)$ (Lemma 1.5(c)) because $p \nmid |F_1|$.

(iii) By (a), $C_{\bar{L}}(\bar{P}) = \bar{P}$ and hence $T_{\bar{L}}(\lambda)$ is not normal in $\bar{L}$ for all $\lambda \in IBr_p(F_1/N)$ because $P/F_1 \cap T_{L/F_1}(\lambda) = E$.

By Lemma 1.5(a), $IBr_p(F_1/N)$ is an irreducible $\bar{G}$-module and therefore a semisimple $\bar{L}$-module. Moreover $IBr_p(F_1/N)$ fulfils our assumption of Lemma 1.9. This implies that $IBr_p(F_1/N)$ is an irreducible $\bar{L}$- and $\bar{P}$-module. Hence F_1/N is a chief factor in P.

2.11. Lemma. *Let G be solvable, $O_p(G) = E$, and $\tau_p(G) = 1$. We put $F_j = F_j(G)$. Suppose that G/F_2 is cyclic of order r^2 and r a prime. ($r \neq p$ by Theorem* 2.4(d).) *Then $r \nmid |F_2/F_1|$ and in particular $r^3 \nmid |G/F_1|$.*

Proof. We put again $\bar{U} := UF_1/F_1$ for all $U \leqslant G$. Further let $\bar{P} \in Syl_p(\overline{F_2})$, $\bar{R} \in Syl_r(\overline{F_2})$, and $\overline{S_j} \in Syl_{s_j}(\overline{F_2})$ for primes $s_j \mid |\overline{F_2}|$ with $r \neq s_j \neq p$ ($j = 1, ..., k$):

$$\overline{F_2} = \bar{P} \times \bar{R} \times \overline{S_1} \times \cdots \times \overline{S_k}.$$

Theorem 2.4(b) forces $|\overline{S_j}| \mid s_j$, $|\bar{P}| \mid p$, and $|\bar{R}| \mid r$. We put $\bar{S} := \overline{S_1} \times \cdots \times \overline{S_k}$. ($\bar{S} = \bar{E}$ possible.)

(i) Assertion. *$F_2 = O_{p'p}(G)$, in particular $C_{\bar{G}}(\bar{P}) = \overline{F_2}$ and $\bar{P} \neq \bar{E}$.*

Proof. By Theorem 2.4(h), $O_{p'p}(G) \cap C_G(\bar{R} \times \bar{S}) = F_2$. Since G/F_2 is cyclic of prime power order we obtain either $O_{p'p}(G) = F_2$ or $C_G(\bar{R} \times \bar{S}) = F_2$. In the last case $|G : C_G(\bar{R} \times \bar{S})| = r^2$, a contradiction to Theorem 2.4(g). Therefore $O_{p'p}(G) = F_2$.

For the remaining steps in the proof let G be a minimal counterexample; thus $\bar{R} > \bar{E}$.

(ii) *$F(G/\Phi(G)) = F(G)/\Phi(G)$ implies $\Phi(G) = E$.*

(iii) Assertion. *If $\overline{S_j} > \bar{E}$, then $|\bar{G} : C_{\bar{G}}(\overline{S_j})| = r$ and $\overline{F_2} \leqslant C_{\bar{G}}(\overline{S_j})$.*

Proof. $|G : C_G(\bar{R} \times \bar{S})|$ is squarefree by Theorem 2.4(g) and especially

$$|\bar{G} : C_{\bar{G}}(\overline{S_j})| \mid r.$$

Assume that $C_{\bar{G}}(\overline{S_j}) = \bar{G}$. Since $(|\overline{S_j}|, |\bar{G}/\overline{S_j}|) = 1$ there exists a normal subgroup $H \trianglelefteq G$ such that

$$\bar{G} = \bar{H} \times \overline{S_j}$$

and we obtain

$$\bar{H}/\overline{F_2(H)} = \bar{H}/(\overline{F_2} \cap \bar{H}) \cong \overline{F_2}\bar{H}/\overline{F_2} = \bar{G}/\overline{F_2}$$

is cyclic of order r^2. (Note: $F_2(G) \cap H = F_2(H)$.) $\overline{F_2} = \overline{S_j} \times \overline{F_2(H)}$ implies further $r \mid |\overline{F_2(H)}|$. Hence $H = G$, in particular $\overline{S_j} = \bar{E}$, because G is a minimal counterexample.

(iv) ASSERTION. *Let $\bar{D} \in Syl_r(\bar{G})$. Then $N_{\bar{G}}(\bar{D}) = \bar{D}$.*

Proof. Assume $N_{\bar{G}}(\bar{D}) > \bar{D}$. Since $\bar{G} = (\bar{P} \times \bar{S})\bar{D}$ there is either $s_j \mid |N_{\bar{G}}(\bar{D})|$ for at least one prime s_j or $p \mid |N_{\bar{G}}(\bar{D})|$ and hence

$$\overline{S_j} \leqslant N_{\bar{G}}(\bar{D}) \qquad \text{for at least one } \overline{S_j} > \bar{E}$$

or

$$\bar{P} \leqslant N_{\bar{G}}(\bar{D}).$$

(Note that $|\bar{F}_2|$ is squarefree.) This implies either $\bar{D} \leqslant C_{\bar{G}}(\overline{S_j})$ $(\overline{S_j} \trianglelefteq \bar{G})$, a contradiction to (iii), or $\bar{D} \leqslant C_{\bar{G}}(\bar{P})$, a contradiction to (i).

By (ii) and a theorem of Gaschütz [2, III, 4.5], we have $F_1 = N_1 \times \cdots \times N_m$ the direct product of minimal normal subgroups of G, resp. a direct sum of irreducible $\bar{G}$-modules (over suitable fields). Also, $IBr_p(F_1)(=Irr(F_1))$ is a direct sum of irreducible $\bar{G}$-modules $(IBr_p(F_1) = IBr_p(N_1 \times \cdots \times IBr_p(N_m))$ and

$$\bigcap_{i=1}^{m} C_{\bar{G}}(IBr_p(N_i)) = C_{\bar{G}}(IBr_p(F_1)) = \bar{E}.$$

By Lemma 1.11 we obtain an irreducible module $W \leqslant IBr_p(F_1)$ with $C_{\bar{R}}(W) = \bar{E}$. Let W be fixed for the rest of this proof.

(v) *By Lemma* 1.8, *$\bar{R}$ operates fixpointfreely on W and $r \nmid |W|$.*

(vi) ASSERTION. *Let $1 \neq \lambda \in W$ and $\overline{B_\lambda} \in Syl_r(T_{\bar{G}}(\lambda))$. Then $\overline{B_\lambda} \cong C_{r^2}$ is cyclic of order r^2. For every Sylow r-subgroup $\bar{D} \in Syl_r(\bar{G})$ there exists a suitable $\overline{B_\lambda}$ such that*

$$\bar{D} = \overline{B_\lambda} \times \bar{R}.$$

Hence the Sylow r-subgroups of $\bar{G}$ are of the form $C_{r^2} \times C$ and in particular $\bar{R} \leqslant Z(\bar{G})$. Moreover, every $\overline{B_\lambda}$ is a complement of $\overline{F_2}$ in $\bar{G}$, in particular $\bar{G}$ splits over $\overline{F_2}$ and we have

$$\bar{G} = T_{\bar{G}}(\lambda) \cdot \overline{F_2}.$$

Proof. By Corollary 1.2, $r^2 \nmid |G : T_G(\lambda)|$ for all $\lambda \in IBr_p(F_1)$ and in particular $r^2 \mid |T_{\bar{G}}(\lambda)|$ because $|\bar{G}|_r = r^3$. Let $1 \neq \lambda \in W$ be fixed chosen. Since $T_{\bar{G}}(\lambda) \cap \bar{R} = \bar{E}$ (v), we obtain

$$\overline{D_0} := \overline{B_\lambda} \cdot \bar{R} \in Syl_r(\bar{G}).$$

$\overline{B_\lambda}$ acts trivial on $\bar{R}$, because $|\bar{R}| = r$ and $\bar{R} \trianglelefteq \bar{G}$. Hence $\overline{D_0} = \overline{B_\lambda} \times \bar{R}$, $|\overline{B_\lambda}| = r^2$ and $\overline{B_\lambda}$ is a complement of $\overline{F_2}$ and cyclic ($\overline{B_\lambda} \cap \overline{F_2} = \bar{E}$).

Let $\bar{D} \in Syl_r(\bar{G})$ be any Sylow r-subgroup of $\bar{G}$. There exists $g \in \bar{G}$ with

$$\bar{D} = \overline{D_0^g} = \overline{B_\lambda^g} \times \overline{R^g} = \overline{B_\lambda^g} \times \bar{R}.$$

Further $\overline{B_\lambda^g} \in Syl_r(T_{\bar{G}}(\lambda^g))$.

(vii) ASSERTION. *Every $\overline{S_j} > \bar{E}$ operates fixpointfreely on W. Hence $\bar{S}$ acts fixpointfreely on W, if $\bar{S} > \bar{E}$.*

Proof. Assume there exists a $1 \neq \lambda \in W$ and a $1 \neq g \in \overline{S_j}$ such that $\lambda^g = \lambda$. We put $\bar{T} := T_{\bar{G}}(\lambda)$. Since $|\overline{S_j}| = s_j$ is a prime, we obtain $\overline{S_j} \leqslant \bar{T}$. By (vi) $\bar{G} = \overline{TF_2}$, hence $\bar{G}/\overline{F_2} \cong \bar{T}/(\bar{T} \cap \overline{F_2})$ and $\bar{T}/(\bar{T} \cap \overline{F_2})$ operates non trivially on $\overline{S_j}$ ((iii)). In particular there exists an $\alpha \in IBr_p(\bar{T})$ with $r \mid \alpha(1)$ (note: $p \nmid |\overline{S_j}|$). As all Sylow subgroups of $\bar{T}$ are cyclic ((vi)), Corollary 1.4 yields a $\beta \in IBr_p(T)$ such that $\beta_{F_1} = \lambda$. This implies $\beta\alpha \in IBr_p(T)$ [3, VII, 9.12b)] with $r \mid (\beta\alpha)(1)$ and $(\beta\alpha)^G \in IBr_p(G)$. Since $\bar{R} \cap \bar{T} = \bar{E}$ we have $r \mid |G : T|$ and therefore $r^2 \mid (\beta\alpha)^G(1)$; contradiction.

(viii) ASSERTION. *$\bar{P}$ acts trivial on W.*

Proof. Assume $C_{\bar{P}}(W) \neq \bar{P}$: Hence $C_{\bar{P}}(W) = \bar{E}$ because $|\bar{P}| = p$. By (v) and (vii) we obtain that W is a faithful irreducible $\bar{G}$-module and $\bar{F}_2$ acts fixpointfreely on it. Step (vi) yields now

$$\bar{G} = T_{\bar{G}}(\lambda) \cdot \overline{F_2} \qquad \text{for all} \quad \lambda \in W.$$

The same argument we used in Lemma 1.9 shows that W is irreducible as an $\overline{F_2}$-module. By Huppert [2, II, 3.11], the following holds: If dim $W = n$ and char $W = q$, we can identify W with $GF(q^n)^+$ and also

$$\overline{F_2} \lesssim GF(q^n)^\times \qquad \text{and} \qquad \bar{G}/\overline{F_2} \lesssim Aut(GF(q^n) : GF(q)).$$

In particular $|\overline{F_2}| = p \cdot s \cdot r \mid q^n - 1$ holds, with $s = |\bar{S}|$. Further $r^2 \mid n$. The main theorem of Galois theory yields the following:

$$GF(q^{n/r^2}) = (\bar{G}/\overline{F_2})^{Fix} \qquad \text{and} \qquad GF(q^{n/r}) = (\bar{H}/\overline{F_2})^{Fix}.$$

For any $U \leqslant Aut(GF(q^n) : GF(q))$ we denote with U^{Fix} the fixfield under the action of U in $GF(q^n)$. By (i), $C_{\bar{G}}(\bar{P}) = \overline{F_2}$ holds and therefore

$$\bar{P} \nleqslant ((\bar{H}/\overline{F_2})^{Fix})^{\times} = GF(q^{n/r})^{\times}.$$

This implies

$$p \nmid q^{n/r} - 1 \qquad \text{and} \qquad p \,\Big|\, \frac{q^n - 1}{q^{n/r} - 1}.$$

Since $\bar{H} = C_{\bar{G}}(\overline{S_j})$ for all $\overline{S_j} > \bar{E}$ ((iii)), we have

$$\overline{S_j} \leqslant ((\bar{H}/\overline{F_2})^{Fix})^{\times} = GF(q^{n/r})^{\times}$$

but

$$\overline{S_j} \nleqslant ((\bar{G}/\overline{F_2})^{Fix})^{\times} = GF(q^{n/r^2})^{\times}.$$

Therefore

$$s_j \,\Big|\, \frac{q^{n/r} - 1}{q^{n/r^2} - 1}$$

for all $\overline{S_j} > \bar{E}$. By (vi), we have $\bar{R} \leqslant Z(\bar{G})$ and hence

$$\bar{R} \leqslant GF(q^{n/r^2})^{\times} = ((\bar{G}/\overline{F_2})^{Fix})^{\times}.$$

This implies $r \mid q^{n/r^2} - 1$ and

$$q^n \equiv q^{n/r} \equiv q^{n/r^2} \equiv 1 \pmod{r}.$$

Therefore

$$\frac{q^n - 1}{q^{n/r} - 1} = 1 + q^{n/r} + \cdots + q^{(n/r)(r-1)} \equiv 0 \pmod{r}$$

and also

$$\frac{q^{n/r} - 1}{q^{n/r^2} - 1} = 1 + q^{n/r^2} + \cdots + q^{(n/r^2)(r-1)} \equiv 0 \pmod{r}.$$

All in all we obtain

$$p \cdot s \cdot r^2 \,\Big|\, \frac{q^n - 1}{q^{n/r} - 1} \cdot \frac{q^{n/r} - 1}{q^{n/r^2 - 1}} = \frac{q^n - 1}{q^{n/r^2} - 1}.$$

To get a contradiction, we show now $(q^n - 1)/(q^{n/r^2} - 1) \leqslant psr$.
$\overline{F_2}$ acts by multiplying on W ($\cong GF(q^n)^+$),

$$\lambda \to a\lambda \qquad (a, \lambda \in GF(q^n)).$$

By (vi), there exists a complement of $\overline{F_2}$ in $\bar{G}$ and every complement $\bar{B}$ is generated by an element ρ such that

$$\lambda\rho = b_\rho \lambda^{q^{n/r^2}} \qquad (b_\rho, \lambda \in GF(q^n)).$$

In particular there is a bijection between $\{\rho\}$ and $\{b_\rho\}$, thus

$$|\{b_\rho\}| = \text{number of complements of } \overline{F_2}.$$

Since $\bar{G} = T_{\bar{G}}(\lambda)\overline{F_2}$ for all $\lambda \in W$ there exists a b_ρ to every $1 \neq \lambda \in W$ such that

$$\lambda = \lambda\rho = b_\rho \lambda^{q^{n/r^2}}$$

and hence

$$\{\lambda^{1 - q^{n/r^2}}\} \subseteq \{b_\rho\}.$$

Especially

$$\begin{aligned} \frac{q^n - 1}{q^{n/r^2} - 1} &= |\{\lambda^{1 - q^{n/r^2}}\}| \\ &\leqslant |\{b_\rho\}| = \text{number of complements of } \overline{F_2}. \end{aligned}$$

For any $\bar{D} \in Syl_r(\bar{G})$ we have $\bar{D} = \bar{R} \times \bar{B}$ and $\bar{B}$ is a complement of $\overline{F_2}$ in $\bar{G}$. For every Sylow r-subgroup there exist r complements $\bar{B}$ ($|\bar{R}| = r$). $N_{\bar{G}}(\bar{D}) = \bar{D}$ ((iv)) implies $|Syl_r(\bar{G})| = ps$ and hence the number of complements is psr. This yields the contradiction:

$$p \cdot s \cdot r^2 \,\Big|\, \frac{q^n - 1}{q^{n/r^2} - 1} \leqslant p \cdot s \cdot r.$$

(ix) Conclusion. As $|\bar{P}| = p$ there exists an irreducible $\bar{G}$-module $V \leqslant IBr_p(F_1)$ such that $C_{\bar{P}}(V) = \bar{E}$ (Lemma 1.11) and $\bar{P}$ acts fixpointfreely on V (Lemma 1.8(b)).

Assume $C_{\bar R}(V)=\bar E$: The steps (v)–(viii) with V instead of W imply that $\bar P$ acts trivial on V; contradiction. Therefore $C_{\bar R}(V)\neq \bar E$ and because of $|\bar R|=r$, $\bar R$ acts trivial on V. In the following we show that $\bar S$ acts trivial on V too. Hence V is an irreducible $\bar G/\overline{RS}$-module:

$$\begin{array}{ll} \bar G & \\ & \big|\; r^2 \\ \overline{F_2} & \\ & \big|\; p \\ \bar R\times\bar S & \\ & \big| \\ \bar E & \\ \multicolumn{2}{l}{V\oplus W\oplus\cdots} \end{array}$$

For every $1\neq\alpha\in V$ and $1\neq\beta\in W$ we define

$$\overline{B_{\alpha\beta}}=T_{\bar G}(\alpha\beta)=T_{\bar G}(\alpha)\cap T_{\bar G}(\beta).$$

By (v) and (viii), $\bar R$ and $\bar S$ act fixpointfreely on W. Since $\bar P$ acts fixpointfreely on V, we have

$$\overline{B_{\alpha\beta}}\cap\overline{F_2}=\bar E \qquad \text{for all} \quad 1\neq\alpha\in V,\ 1\neq\beta\in W.$$

$\tau_p(G)=1$ implies now $r^2\nmid|\bar G:\overline{B_{\alpha\beta}}|$ and because of $|\bar G|_r=r^3$, we obtain $r^2\,|\,|\overline{B_{\alpha\beta}}|$. Hence $|\overline{B_{\alpha\beta}}|=r^2$ and in particular $\overline{B_{\alpha\beta}}$ is a complement of $\overline{F_2}$ in $\bar G$. As $C_{\bar R}(V)=\bar R$, we obtain

$$\overline{B_{\alpha\beta}}\times\bar R\leqslant T_{\bar G}(\alpha) \qquad \text{for all} \quad 1\neq\alpha\in V,\ 1\neq\beta\in W.$$

Further, $\overline{B_{\alpha\beta}}\times\bar R\in Syl_r(\bar G)$.

Let $1\neq\beta_0\in W$ be fixed chosen. Steps (v), (vii), and (viii) yield

$$T_{\bar G}(\beta_0)=\bar P\cdot\overline{B_{\alpha\beta_0}} \qquad \text{for all} \quad 1\neq\alpha\in V.$$

This implies that for all $1\neq\alpha\in V$ the $\overline{B_{\alpha\beta_0}}$ are conjugate under $\bar P$ because $\overline{B_{\alpha\beta_0}}$ is a Sylow r-subgroup of $T_{\bar G}(\beta_0)$.

Assume $C_{\bar S}(V)\neq\bar S$: Then, there exists an $1\neq\alpha_0\in V$ and an $\overline{S_i}>\bar E$ such that $\overline{S_i}\nleqslant T_{\bar G}(\alpha_0)$. We put

$$\overline{D_0}:=\overline{B_{\alpha_0\beta_0}}\times\bar R.$$

$\bar D_0\leqslant T_{\bar G}(\alpha_0)$ and $\overline{D_0}\in Syl_r(\bar G)$ holds. Let $1\neq g\in\overline{S_i}$. Then

$$\overline{D_1}:=\overline{B_{\alpha_0^g\beta_0}}\times\bar R\leqslant T_{\bar G}(\alpha_0^g)=T_{\bar G}(\alpha_0)^g.$$

Since all $\overline{B_{\alpha\beta_0}}$ are conjugate under $\bar{P}$ for all $1 \neq \alpha \in V$, there exists an $h \in \bar{P}$, such that

$$(\overline{B_{\alpha_0\beta_0}})^h = \overline{B_{\alpha_0^g\beta_0}}$$

and hence

$$\overline{D_1} = \overline{B_{\alpha_0^g\beta_0}} \times \bar{R} = (\overline{B_{\alpha_0\beta_0}})^h \times \bar{R} = (\overline{B_{\alpha_0\beta_0}})^h \times \overline{R^h} = \overline{D_0^h}.$$

This shows

$$\overline{D_0^h} \leqslant T_{\bar{G}}(\alpha_0)^g$$

resp.

$$\overline{D_0^{hg^{-1}}} \leqslant T_{\bar{G}}(\alpha_0).$$

As $\langle g \rangle = \overline{S_i} \trianglelefteq \bar{G}$ and $\langle h \rangle = \bar{P} \trianglelefteq \bar{G}$, if $h \neq 1$, we obtain

$$\overline{D_0} \leqslant \langle \overline{D_0}, \overline{D_0^{hg^{-1}}} \rangle \leqslant \bar{P} \cdot \overline{S_i} \cdot \overline{D_0}.$$

Since $N_{\bar{G}}(\overline{D_0}) = \overline{D_0}$ ((iv)) we have $\overline{D_0} \neq \langle \overline{D_0}, \overline{D_0^{hg^{-1}}} \rangle$. As $|\overline{PS_i D_0} : \overline{D_0}| = ps_i$ and p, s_i are primes, either $\bar{P}$ or $\overline{S_i}$ (or $\overline{PS_i}$) lies in $\langle \overline{D_0}, D_0^{hg^{-1}} \rangle$ and hence in $T_{\bar{G}}(\alpha_0)$. But $\overline{S_i} \nleqslant T_{\bar{G}}(\alpha_0)$ holds by the choice of α_0 and $\bar{S}_i$. Therefore $\bar{P} \leqslant T_{\bar{G}}(\alpha_0)$. This is a contradiction because $\bar{P}$ acts fixpointfreely on V.

Hence $\bar{S}$ acts trivial on V and V is a faithful irreducible $\bar{G}/\overline{RS}$-module because $C_{\bar{R}}(V) = \bar{R}$. $\bar{G}/\overline{RS}$ is a Frobenius group with kernel $\overline{F_2}/\overline{RS} \cong \bar{P}$ and cyclic complement $\bar{G}/\overline{F_2}$ of order r^2. As we mentioned above, $\overline{B_{\alpha\beta}} \leqslant T_{\bar{G}}(\alpha)$ is a complement of $\overline{F_2}$ in $\bar{G}$ (for all $1 \neq \alpha \in V$ and $1 \neq \beta \in W$) and hence

$$\bar{G}/\overline{RS} = (T_{\bar{G}/\overline{RS}}(\alpha)) \cdot \overline{F_2}/\overline{RS} \qquad \text{for all} \quad \alpha \in V.$$

Theorem 2.8 yields now $|\bar{G}/\overline{F_2}| = r$; contradiction.

By proving our main Theorem 2.6 we will consider a minimal counterexample. To get the proof clear we consider at first the structure of a minimal counterexample.

2.12. LEMMA. *Let G be solvable and minimal with $O_p(G) = E$, $\tau_p(G) = 1$, and $\tau(G) \geqslant 3$. If r is a prime with $r^3 \mid \chi(1)$ for a $\chi \in Irr(G)$ then the following hold*:

(a) $|G/F_2(G)| = r^2$;

(b) $F_2(G) = O_{p'p}(G)$ *and* $G/F_2(G)$ *is cyclic*;

(c) $r \nmid |F_2(G)/F_1(G)|$.

Proof. Proposition: We put $F_j := F_j(G)$ and $\bar{U} = UF_1/F_1$ for all $U \leqslant G$. Let $\bar{P} \in Syl_p(\overline{F_2})$. Assume that $\bar{P} = \bar{E}$: Since $O_p(G) = E$ we obtain $G = O_{p'p}(G)$ (Theorem 2.4(c)) and because the Sylow p-subgroups of G are abelian (Theorem 2.4(d)), Lemma 1.13 yields $cd_p(G) = cd(G)$, in particular $\tau(G) = 1$. Hence $\bar{P} \neq \bar{E}$ and $|\bar{P}| = p$ (Theorem 2.4(b)).

Let $\chi \in \mathrm{Irr}(G)$ with $r^3 | \chi(1)$ (r a prime) and $\chi_{F_2} = \psi_1 + \cdots + \psi_k$ with $\psi_j \in Irr(F_2)$. By Lemma 1.13, we have $\tau_p(F_2) = \tau(F_2) = 1$ and so

$$r^2 \mid |G/F_2|.$$

Let $H/F_2 \in Syl_r(G/F_2)$. Then $\chi_H = \varphi + \cdots$ with $r^3 | \varphi(1)$ and $\varphi \in Irr(H)$. Since G is minimal we obtain $H = G$ and G/F_2 is an r-group.

(a) *Assume* $r^3 \mid |G/F_2|$:

(i) ASSERTION. $|G/F_2| = r^3$ *and there exists a* $\chi \in Irr(G/F_1)$ *with* $r^3 = \chi(1)$.

Proof. Let M be a normal subgroup of G such that $|M/F_2| = r^3$ (G/F_2 is abelian (Theorem 2.4(c))). By Lemma 1.1 of Huppert and Manz [5] there exists a $\chi \in Irr(M/F_1)$ with $\chi(1) = r^3$. The minimal choice of G forces $G = M$.

(ii) ASSERTION. F_1 *is a minimal normal subgroup of* G.

Proof. Lemma 1.11 yields a normal subgroup $N \trianglelefteq G$ such that F_1/N is a chief factor in G and $C_P(F_1/N) \leqslant F_1$. Especially, $O_p(G/N)$ is trivial. By step (i), we obtain a $\chi \in Irr(G/N)$ with $r^3 | \chi(1)$. Since G is minimal, $N = E$ holds.

(iii) ASSERTION. $\overline{F_2}$ *acts fixpointfreely on* F_1 *and on* $IBr_p(F_1)$ *also.*

Proof. As $C_G(F_1) = F_1$ and F_1 is a minimal normal subgroup, every Sylow subgroup of $\overline{F_2}$ acts fixpointfreely on F_1 (Lemma 1.8(b)). (Note that $\overline{F_2}$ is cyclic of squarefree order.) Hence $\overline{F_2}$ acts fixpointfreely on F_1 and because of $p \nmid |F_1|$, on $IBr_p(F_1)$ too (Lemma 1.5(c)).

(iv) ASSERTION. *There exists a normal subgroup* $H \trianglelefteq G$ *such that*

$$G/F_2 = H/F_2 \times O_{p'p}(G)/F_2$$

and H/F_2 *is cyclic with* $r^2 \mid |H/F_2|$.

Proof. If $O_{p'p}(G) = F_2$ the assertion follows by $H = G$ and

Theorem 2.4(f). Suppose $O_{p'p}(G) > F_2$. The statements (g) and (h) of Theorem 2.4 yield a normal subgroup $H \trianglelefteq G$ such that

$$r^2 \nmid |G/H| \qquad \text{and} \qquad O_{p'p}(G) \cap H = F_2.$$

As $|G/F_2/ = r^3$ we obtain $r^2 = |H/F_2|$ and hence

$$G = O_{p'p}(G) \cdot H$$

resp.

$$G/F_2 = H/F_2 \times O_{p'p}(G)/F_2.$$

Also, $H/F_2 \cong G/O_{p'p}(G)$ is cyclic (Theorem 2.4(f)).

As $p \nmid |F_1|$ and F_1 is a minimal normal subgroup of G, $IBr_p(F_1)$ is a faithful irreducible $\bar{G}$-module (over a suitable finite field). $\tau_p(G) = 1$ implies for every $\lambda \in IBr_p(F_1)$ that $|\bar{G} : T_{\bar{G}}(\lambda)|$ is squarefree.

(v) Assertion. *$O_{p'p}(G) = F_2$ and hence G/F_2 is cyclic.*

Proof. Assume $O_{p'p}(G) > F_2$: By (iv), we obtain

$$G/F_2 \cong C_{r^2} \times C_r.$$

Especially, $\Phi(G/F_2)$ is not trivial. We define

$$K/F_2 := \Phi(G/F_2).$$

As mentioned above, $r^2 \nmid |\bar{G} : T_{\bar{G}}(\lambda)|$ for all $\lambda \in IBr_p(F_1)$. Therefore we obtain either $T_{\bar{G}}(\lambda) \cdot \overline{F_2} = \bar{G}$ or $(T_{\bar{G}}(\lambda)\overline{F_2})/\overline{F_2}$ is a maximal subgroup of $\bar{G}/\overline{F_2}$. In any case

$$\bar{K} \leqslant T_{\bar{G}}(\lambda) \cdot \overline{F_2} \text{ hence } \bar{K} = T_{\bar{K}}(\lambda) \cdot \overline{F_2} \qquad \text{for all} \quad \lambda \in IBr_p(F_1).$$

Since $\overline{F_2}$ acts fixpointfreely on $IBr_p(F_1)$, we have

$$T_{\bar{K}}(\lambda) \cap \overline{F_2} = \bar{E} \qquad \text{for all} \quad \lambda \in IBr_p(F_1).$$

Further, $T_{\bar{K}}(\lambda)$ is not normal in $\bar{K}$ for all $\lambda \in IBr_p(F_1)$ because of $\overline{F_2} = F(\bar{K})$ and $T_{\bar{K}}(\lambda) \cap \overline{F_2} = \bar{E}$. By Clifford theory, $IBr_p(F_1)$ is a semisimple $\bar{K}$-module and by Lemma 1.9 even irreducible and irreducible as a $\overline{F_2}$-module too. Huppert [2, II, 3.11] yields that $\bar{G}$ is isomorphic to a group of semilinear maps over a finite field. Since $C_{\bar{G}}(\overline{F_2}) = \overline{F_2}$, $\bar{G}/\overline{F_2}$ is isomorphic to a subgroup of the automorphism group and in particular cyclic; contradiction.

(vi) Assertion. $r \nmid |F_2/F_1|$.

Proof. As G/F_2 is cyclic ((v)), there exists a unique normal subgroup $K \trianglelefteq G$ with $|K/F_2| = r^2$. Lemma 2.11 yields $r \nmid |F_2/F_1|$.

(vii) CONCLUSION OF (a). By (i), (v), and (vi), G fulfils all conditions of Lemma 2.10:

(1) $F_2 = O_{p'p}(G)$,
(2) $|G/F_2| = r^3$, and
(3) $r \nmid |F_2/F_1|$.

By Lemma 2.10(a), there exist a normal subgroup $\bar{L} \trianglelefteq \bar{G}$ with $\bar{P} \leqslant \bar{L}$ and $C_{\bar{L}}(\bar{P}) = \bar{P}$. Further, $|\bar{L}/\bar{P}| = r^2$ by construction. Part (b) of 2.10 and (ii) yield that F_1 is a chief factor in P and hence a chief factor in L. For all $\lambda \in IBr_p(F_1)$

$$\bar{L} = T_{\bar{L}}(\lambda) \cdot \bar{P} \qquad \text{(Lemma 2.10(b)(i))}$$

and Theorem 2.8 yields $|\bar{L}/\bar{P}| = r$ a prime; contradiction.

(b) Lemma 1.13 yields $\tau(O_{p'p}(G)) = \tau_p(O_{p'p}(G)) = 1$. If $\chi \in Irr(G)$ with $r^3 \mid \chi(1)$ we obtain

$$\chi_{O_{p'p}(G)} = \psi_1 + \cdots + \psi_k \qquad \text{with} \quad r^2 \nmid \psi_j(1) \; (\psi_j \in Irr(O_{p'p}(G))).$$

In particular $r^2 \mid |G : O_{p'p}(G)|$. $F_2 \leqslant O_{p'p}(G)$ and $|G/F_2| = r^2$ implies $F_2 = O_{p'p}(G)$. Hence G/F_2 is cyclic (Theorem 2.4(f)).

(c) Since G/F_2 is cyclic of order r^2, Lemma 2.11 yields $r \nmid |F_2/F_1|$.

2.13. PROOF OF MAIN THEOREM 2.6. Let G be a minimal counterexample; hence G is solvable, $O_p(G) = E$ and $\tau_p(G) = 1$, but $\tau(G) \geqslant 3$. We put again $F_j := F_j(G)$ and $\bar{U} := (UF_1)/F_1$ for all $U \leqslant G$. Since G is a minimal counterexample, there exists a $\chi \in Irr(G)$ with $r^3 \mid \chi(1)$ for a prime r. Lemma 2.12 yields now

(1) $O_{p'p}(G) = F_2$,
(2) G/F_2 is cyclic of order r^2, and
(3) $r \nmid |F_2/F_1|$.

We use the same notation as in Lemma 2.11:

$$\bar{P} \in Syl_p(\overline{F_2}) \qquad \text{and} \qquad \overline{S_j} \in Syl_{s_j}(\overline{F_2}) \qquad \text{for primes } s_j \neq p \; (j = 1, ..., k).$$

As $|\overline{F_2}|$ is squarefree (Theorem 2.4(b)), we know $|\overline{S_j}| \mid s_j$ and $|\bar{P}| = p$ because of $O_{p'p}(G) = F_2$. Further let $\bar{S} := \overline{S_1} \times \cdots \times \overline{S_k}$.

(i) ASSERTION. *$r = |\bar{G} : C_{\bar{G}}(\bar{S}_j)|$ for all $\bar{S}_j > \bar{E}$ ($j = 1, ..., k$). This implies $|C_{\bar{G}}(\bar{S}_j)|_r = r$ because $|\bar{G}|_r = r^2$.*

Proof. By Theorem 2.4(g), $|\bar{G} : C_{\bar{G}}(\bar{S})|$ is squarefree, in particular $|\bar{G} : C_{\bar{G}}(\bar{S}_j)| \mid r$ ($j = 1, ..., k$). Assume $C_{\bar{G}}(\bar{S}_j) = \bar{G}$ for any $\bar{S}_j > \bar{E}$. Since $(|\bar{G}/\bar{S}_j|, |\bar{S}_j|) = 1$ ($r \nmid |\bar{S}|$), there exists an $H \trianglelefteq G$ such that $\bar{G} = \bar{H} \times \bar{S}_j$. $|G/H| = s_j \neq r$ yields

$$\chi_H = \psi + \cdots \qquad \text{with } r^3 \mid \psi(1)\ (\psi \in Irr(H)).$$

The minimality of G forces $H = G$; contradiction.

(ii) ASSERTION. *If $\gamma \in IBr_p(F_1)$ with $r \mid \gamma(1)$, then $r \nmid |G : T_G(\gamma)|$. Hence $r^2 \mid |T_{\bar{G}}(\gamma)|$ because $|\bar{G}|_r = r^2$. Further there exists a $\beta \in IBr_p(T_G(\gamma))$ such that*

$$\beta_{F_1} = \gamma.$$

Proof. Since $\tau_p(G) = 1$ and $r \mid \gamma(1)$ we have $r \nmid |\bar{G} : T_{\bar{G}}(\gamma)|$ and $|T_{\bar{G}}|_r = |\bar{G}|_r = r^2$. As $\bar{G}/\bar{F}_2$ and $\bar{F}_2$ are cyclic, $\bar{G}$ has only cyclic Sylow subgroups (note that $(|\bar{G}/\bar{F}_2|, |\bar{F}_2|) = 1$) and by Corollary 1.4 we can continue γ to $T_G(\gamma)$.

(iii) ASSERTION. *If $\gamma \in IBr_p(F_1)$ with $r \mid \gamma(1)$ and $|T_{\bar{G}}(\gamma)| = m \cdot p \cdot r^2$, then $m = 1$.*

Proof. We put $T := T_G(\gamma)$. By (ii) there exists a $\beta \in IBr_p(T)$ such that $\beta_{F_1} = \gamma$. *Assume $m > 1$*: This forces $s_j \mid |\bar{T}|$ for at least one $j \in \{1, ..., k\}$. In particular

$$\bar{E} \neq \bar{S}_j \leqslant \bar{T}.$$

As $|\bar{T}|_r = |\bar{G}|_r = r^2$, step (i) yields $r \mid |\bar{T} : C_{\bar{T}}(\bar{S}_j)|$ and because of $p \nmid |\bar{S}_j|$, there exists a $\delta \in IBr_p(\bar{T})$ with $r \mid \delta(1)$. By Huppert and Blackburn [3, VII, 9.12(b)] we have $\beta\delta \in IBr_p(T)$ and $(\beta\delta)^G \in IBr_p(G)$ (Lemma 1.1). But $r^2 \mid (\beta\delta)(1)$ yields a contradiction to $\tau_p(G) = 1$.

(iv) ASSERTION. *There exists a $\gamma \in IBr_p(F_1)$ with $\gamma(1) = r$ and $|T_{\bar{G}}(\gamma)| = p \cdot r^2$.*

Proof. If $\chi_{F_1} = \psi + \cdots$ with $\psi \in Irr(F_1)$, then $r \mid \psi(1)$ because $|\bar{G}|_r = r^2$ and $r^3 \mid \chi(1)$. By Huppert [2, V, 17.11(a), (b)], there exists a $\varphi \in Irr(T_G(\psi))$ such that $\chi = \varphi^G$ and $\varphi_{F_1} = e \cdot \psi$. Since $p \nmid |F_1|$ we can identify ψ with a Brauer character of F_1. Assume $p \nmid |T_G(\psi)|$: Then φ is an irreducible Brauer character of $T_G(\psi)$ and by Lemma 1.1 φ^G is an irreducible Brauer character. But $r^3 \mid (\varphi)^G(1)$ is a contradiction to $\tau_p(G) = 1$. Hence $p \mid |T_{\bar{G}}(\psi)|$.

Steps (ii) and (iii) yield now $|T_{\bar{G}}(\psi)| = pr^2$. Since F_1 is nilpotent, we can write ψ as

$$\psi = \gamma\alpha \qquad \text{with} \quad \gamma \in IBr_p(O_r(F_1)), \alpha \in IBr_p(O_{r'}(F_1)).$$

This forces $\gamma(1) = r$ $(\tau_p(F_1) = \tau(F_1) = 1)$. Since $T_G(\psi) = T_G(\gamma) \cap T_G(\alpha)$, we have $r^2 p \mid |T_{\bar{G}}(\gamma)|$ and by (iii), $|T_{\bar{G}}(\gamma)| = pr^2$.

(v) CONCLUSION. By Lemma 2.10(b), there exists a normal subgroup $N \trianglelefteq G$ such that $N \leqslant F_1$ and F_1/N is a chief factor in P. Further, $C_{\bar{P}}(F_1/N) = \bar{E}$.

Let $\gamma \in IBr_p(F_1)$ with $\gamma(1) = r$ and $|T_{\bar{G}}(\gamma)| = p \cdot r^2$ ((iv)). In particular, $P \leqslant T_G(\gamma)$. As $p \nmid |F_1|$ and F_1/N is a chief factor in P, Isaacs [6, 6.18] yields one of the following three cases:

— $\gamma_N = \gamma_0 \in IBr_p(N)$;
— $\gamma_N = e \cdot \lambda$, $\lambda \in IBr_p(N)$ and $e^2 = |F_1/N|$;
— $\gamma_N = \sum_{i=1}^{t} \lambda_i$, $\lambda_i \in IBr_p(N)$ and $t = |F_1/N|$.

If γ_N is not irreducible, both last cases yield

$$r^2 = |F_1/N| \qquad \text{or} \qquad r = |F_1/N|$$

because $\gamma(1) = r$. Since $\bar{P}$ acts faithfully on F_1/N we obtain

$$p \mid r^2 - 1.$$

Conversely, we have $|G/F_2| = |G/O_{p'p}(G)| = r^2$ and Theorem 2.4(f) yields

$$r^2 \mid p - 1;$$

contradiction. Hence $\gamma_N = \gamma_0 \in IBr_p(N)$.

We show now $T_G(\gamma) = T_G(\gamma_0)$: $T_G(\gamma) \leqslant T_G(\gamma_0)$ is clear.

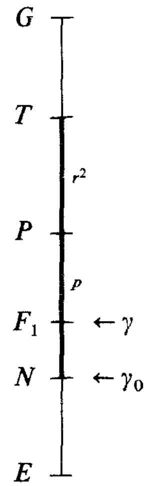
Let $g \in T_G(\gamma_0)$. Then $\gamma_N^g = \gamma_0 = \gamma_0^g$ and a theorem of Gallagher (Isaacs [6, 6.17]) yields a uniquely determined $\lambda \in IBr_p(F_1/N)$ such that

$$\gamma^g = \gamma\lambda.$$

As $P \trianglelefteq G$ and $P \leqslant T_G(\gamma)$, we have $P \leqslant T_G(\gamma^g) = T_G(\gamma)^g$ too. For all $h \in P$ follows:

$$\gamma\lambda = \gamma^g = (\gamma^g)^h = \gamma^h \lambda^h = \gamma\lambda^h.$$

Since λ is uniquely determined, we obtain $\lambda^h = \lambda$ for all $h \in P$. By Lemmas 1.8(b) and 1.5(c) $\bar{P}$ acts fixpointfreely on $IBr_p(F_1/N)$; therefore $\lambda = 1$ and hence $\gamma^g = \gamma$. This shows $T_G(\gamma_0) \leqslant T_G(\gamma)$. We put $T := T_G(\gamma) = T_G(\gamma_0)$.



By step (ii), there exists a $\beta \in IBr_p(T)$ such that $\beta_{F_1} = \gamma$, resp. $\beta_N = \gamma_0$. We show now the existence of an $\alpha \in IBr_p(T/N)$ with $r \mid \alpha(1)$: F_1/N is a minimal normal subgroup of T/N because F_1/N is a chief factor in P and $N \trianglelefteq G$. Further, $C_{P/N}(F_1/N) = F_1/N$. Since $O_{p'p}(G) = F_2$ and $r \nmid |F_2/F_1|$ we have $r \nmid |C_{\bar{G}}(\bar{P})|$ and hence

$$C_{T/F_1}(P/F_1) = P/F_1.$$

We have further $T/P \cong G/O_{p'p}(G) \cong C_{r^2}$ and Corollary 2.9 yields an $\alpha \in IBr_p(T/N)$ with $r \mid \alpha(1)$. This implies $\beta\alpha \in IBr_p(T)$ [3, VII, 9.12(b)] and because of $T = T_G(\gamma_0)$, Lemma 1.1 shows $(\beta\alpha)^G \in IBr_p(G)$. Finally, $r^2 \mid (\beta\alpha)^G(1)$ yields a contradiction to $\tau_p(G) = 1$.

Easy examples show that the bound in Theorem 2.6 is best as possible.

2.14 Examples. (a) Let $G = GL(2, 3)$. Then

$$cd_3(G) = \{1, 2, 3\} \qquad \text{and} \qquad cd(G) = \{1, 2, 3, 4\}.$$

(b) Let p, q, r be primes, $n \in \mathbb{N}$, and

$$p = \frac{q^n - 1}{q^{n/r} - 1}.$$

For instance $(p, q, r, n) = (5, 2, 2, 4)$; $(7, 2, 3, 3)$; $(13, 3, 3, 3)$; $(757, 3, 3, 9)$; .... We put

$$P := GF(q^n)^{(1 - q^{n/r})} = \{v^{(1 - q^{n/r})} \mid v \in GF(q^n)^\times\}.$$

Hence

$$|P| = \frac{q^n - 1}{q^{n/r} - 1} = p.$$

Further let $V = GF(q^n)^+$ and P operates by multiplications on V. ($v \to av$, with $a \in P$, $v \in V$). If σ is the Frobenius automorphism of $GF(q^n)$ ($v\sigma = v^q$ for $v \in V$), we put

$$\alpha = \sigma^{n/r}; \qquad \text{hence} \quad v\alpha = v^{q^{n/r}} \text{ and } |\langle \alpha \rangle| = r.$$

Further let $K := P\langle \alpha \rangle$ be the semidirect product of P with $\langle \alpha \rangle$ and K acts by the natural way on V. Every complement R of P in K is generated by a uniquely determined element ρ of the form

$$v\rho = b_\rho v^{q^{n/r}} \qquad (v \in V, b_\rho \in P).$$

In particular

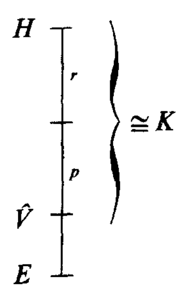
$$|\{b_\rho\}| = p = \text{number of complements of } P \text{ in } K.$$

Since $b_\rho \in P$, we obtain $\{b_\rho\} = P$ and to every $v \in V \backslash \{0\}$ exists a $b_\rho = v^{(1 - q^{n/r})} \in P$ such that

$$v = b_\rho v^{q^{n/r}} = v\rho.$$

This implies $|C_K(v)| = r$ for every $0 \neq v \in V$.

Consider now the action of K on $\hat{V} = Irr(V)$ and put $H = \hat{V} \cdot K$:



As everybody knows $Irr(\hat{V}) \cong_H V$ and we obtain for ever $\lambda \in Irr(\hat{V})$

$$|T_K(\lambda)| = r.$$

It is easy to see that

$$cd(H) = \{1, p, r\} \qquad \text{and} \qquad cd_p(H) = \{1, p\}.$$

If L is an r-group with $cd(L)=\{1, r\}$, we put $G:=H\times L$ and obtain

$$cd(G)=\{1, p, r, pr, r^2\} \qquad \text{and} \qquad cd_p(G)=\{1, p, r, pr\}.$$

2.15. *Remark.* If p is a prime of the form

$$p=\frac{q^n-1}{q^{n/r}-1} \qquad (p, q, r) \text{ primes}, n\in\mathbb{N}),$$

we can construct groups G with $\tau_p(G)=1$ and $\tau(G)=2$ (see Example 2.14(b)). If p is any prime, not necessarily of the form above, we do not know such examples.

The set of these primes contains the Fermat and Mersenne primes:

$$p=\frac{2^{2^n}-1}{2^{2^{n-1}}-1}=2^{2^{n-1}}+1$$

resp.

$$p=2^r-1 \ (r \text{ a prime}).$$

3. A Generalization

In the last section we considered the case that all Brauer characters are squarefree. Since this restriction is very strong, it was possible to describe the group very precisely. In this section we generalize it by assuming $\tau_p(G)\leqslant n$ ($n\in\mathbb{N}$). Remembering the definition of $\tau_p(G)$, it is easy to see that we cannot expect strong results. If $n\in\mathbb{N}$ is very large, the assumption $\tau_p(G)\leqslant n$ is nearly no restriction to the structure of G. Nevertheless we can find a bound of $\tau(G)$ depending on $\tau_p(G)$. Furthermore it is possible to bound the derived length of G.

3.1. Lemma. *Let G be solvable with $\Phi(G)=E$ and $|G/F(G)|=r^a$ for a prime r ($a\in\mathbb{N}$). Further let $|O_p(G)|=p^f$ ($f\in\mathbb{N}_0$) and $S\leqslant F(G)$ the $\{r, p\}'$-Hall subgroup of $F(G)$, in particular $S\trianglelefteq G$. As $\Phi(G)=E$, $F(G)$ is abelian and hence $F(G)\leqslant C_G(S)$. The following assertions hold*:

(a) $|G/C_G(S)| \mid r^{(2\tau_p(G))}$.

(b) *If $r=p$ or $O_p(G)=E$, then* $|G/F(G)| \mid r^{(2\tau_p(G))}$.

(c) $|C_G(S)/F(G)| \mid r^{2f(\log_2 p)}$.

Proof. (a) By Passman [9, Cor. 2.4.(iii)] there exists an $h \in S$ with

$$|C_G(h)/C_G(S)| \leqslant |G/C_G(S)|^{1/2}.$$

As S is abelian and $(|G/C_G(S)|, |S|) = 1$, S and $Irr(S)$ are permutation isomorphic as $G/C_G(S)$-set (Isaacs [6, 13.24(b)] and there exists a $\lambda \in Irr(S)$ such that

$$|T_G(\lambda)/C_G(S)| \leqslant |G/C_G(S)|^{1/2}$$

resp.

$$|G : T_G(\lambda)| \geqslant |G/C_G(S)|^{1/2}.$$

Since $p \nmid |S|$, we have $Irr(S) = IBr_p(S)$ and Corollary 1.2 yields

$$|G/C_G(S)|^{1/2} \leqslant |G : T_G(\lambda)| \mid r^{\tau_p(G)}.$$

(b) By Lemma 1.10, $G/F(G)$ acts faithfully on the r'-part of $F(G)$. If $r = p$ or $O_p(G) = E$, then S is the r'-part of $F(G)$ and the assertion follows by (a).

(c) Let w.l.o.g. $r \neq p$ and $O_p(G) \neq E$ (proof of (b)). By Lemma 1.10, $G/F(G)$ acts faithfully on $O_p(G) \times S$ and hence $C_G(S)/F(G)$ acts faithfully on $O_p(G)$. We put $H := C_G(S)$. Passmann [9, Cor.2.4(iii)] yields a $g \in O_p(G)$ such that

$$|C_{H/F(G)}(g)| \leqslant |H/F(G)|^{1/2}$$

resp.

$$|H/F(G)|^{1/2} \leqslant |H/F(G) : C_{H/F(G)}(g)| \leqslant |O_p(G)| - 1 = p^f - 1.$$

If $|H/F(G)| = r^b$, then $r^b < p^{2f}$. Hence

$$b < \log_r p^{2f} \leqslant 2f(\log_2 p).$$

3.2. THEOREM. *Let G be solvable and $O_p(G) = E$. Then the following hold*:

(a) $n(G) \leqslant 2\tau_p(G) + 4$;

(b) $dl(G) \leqslant 3\tau_p(G) + \log_2(\tau_p(G)) + 5$;

(c) $\tau(G) \leqslant \tau_p(G)^2\,(8 \log_2 p + 4) + \tau_p(G)(8 \log_2 p + 7)$.

Proof. We put $F_j = F_j(G)$, $\Phi_j = \Phi_j(G)$, and $\tau_p = \tau_p(G)$. $(E = F_0 \leqslant \Phi_1 < F_1 \leqslant \Phi_2 < F_2 \leqslant \cdots \leqslant \Phi_n < F_n = G.)$

(a), (b) Let $q \mid |F_2/F_1|$ be any prime divisor and $Q/F_1 \in Syl_q(F_2/F_1)$. Considering the group Q/Φ_1, Lemma 3.1(b) yields

$$|Q/F_1| \mid q^{2\tau_p}.$$

(Note: $p \nmid |F_1|$ and $\tau_p(Q) \leqslant \tau_p(G)$.) By a theorem of Gaschütz [2, III, 4.5],

$$F_2/\Phi_2 = F(G/\Phi_2) = V_1 \oplus \cdots \oplus V_k$$

is a direct sum of irreducible G/F_2-modules (over suitable fields) and G/F_2 acts faithfully on $V_1 \oplus \cdots \oplus V_k$ [2, III, 4.2(b)]. Hence

$$G/F_2 = G\Bigg/\left(\bigcap_{i=1}^{k} C_G(V_i)\right) \lesssim G/C_G(V_1) \times \cdots \times G/C_G(V_k).$$

Let char $V_i = q_i$. Then $G/C_G(V_i)$ is isomorphic to an irreducible subgroup of $GL(\dim V_i, q_i)$. Theorem 2.5(b) of Leisering and Manz [7] yields now

$$dl(G/C_G(V_i)) \leqslant \dim V_i + 2.$$

Since $|Q/F_1| \mid q^{2\tau_p}$ for all Sylow q-subgroups of F_2/F_1 we obtain $\dim V_i \leqslant 2\tau_p$ $(i = 1, ..., k)$. Hence $dl(G/F_2) \leqslant 2\tau_p + 2$ and assertion (a) follows. We show now $dl(F_2/F_1) \leqslant 2 + \log_2 \tau_p$ and $dl(F_1) \leqslant \tau_p + 1$. As $|Q/F_1| \mid q^{2\tau_p}$ for all Sylow q-subgroups of F_2/F_1, we obtain by a theorem of P. Hall [2, III, 7.11]

$$dl(Q/F_1) \leqslant 1 + \log_2(2\tau_p) = 2 + \log_2 \tau_p.$$

Let R be any Sylow subgroup of F_1. As $p \nmid |R|$, the theorem of Taketa yields [2, V, 18.6]

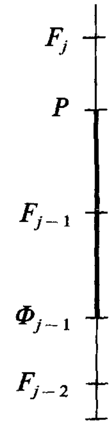
$$dl(R) \leqslant \tau_p(R) + 1 \leqslant \tau_p + 1$$

and hence $dl(F_1) \leqslant \tau_p + 1$. Therefore

$$\begin{aligned} dl(G) &\leqslant dl(G/F_2) + dl(F_2/F_1) + dl(F_1) \\ &\leqslant 2\tau_p + 2 + \log_2 \tau_p + 2 + \tau_p + 1 \\ &= 3\tau_p + \log_2 \tau_p + 5. \end{aligned}$$

(c) (i) Assertion. $|F_j/F_{j-1}|_p \mid p^{2\tau_p}$ *for all* $j = 1, ..., n(G)$.

Proof. Let $j \in \{2, \ldots, n(G)\}$ and $P/F_{j-1} \in Syl_p(F_j/F_{j-1})$. We consider P/Φ_{j-1}:



Since $F(P/\Phi_{j-1}) = F_{j-1}/\Phi_{j-1}$, Lemma 3.1(b) yields

$$|P/F_{j-1}| \mid p^{2\tau_p}.$$

If $j = 1$, the assertion is obvious because $p \nmid |F_1|$.

(ii) Assertion. *Let r be a prime and $|G/F_1|_r = r^m$. Then*

$$m \leqslant (n(G)-2)(4\tau_p \log_2 p + 2\tau_p) + 2\tau_p.$$

Proof. If $r = p$, the assertion follows by (i). We consider F_j/Φ_{j-1} for $j \in \{3, \ldots, n(G)\}$. By (i), we have

$$|O_p(F_j/\Phi_{j-1})| = |F_{j-1}/\Phi_{j-1}|_p \mid p^{2\tau_p}.$$

Let $R/F_{j-1} \in Syl_r(F_j/F_{j-1})$. Lemma 3.1(a), (c) yields

$$|R/F_{j-1}| \mid r^{(2\tau_p + 4\tau_p \log_2 p)}.$$

In case $j = 2$ even

$$|R/F_1| \mid r^{(2\tau_p)}$$

follows by Lemma 3.1(b). Hence

$$m \leqslant (n(G)-2)(4\tau_p \log_2 p + 2\tau_p) + 2\tau_p.$$

(iii) Assertion. $\tau(G) \leqslant (n(G)-2)(4\tau_p(G) \log_2 p + 2\tau_p(G)) + 3\tau_p(G)$. (W.l.o.g. $n(G) \geqslant 2$. *Otherwise we have* $\tau_p(G) = \tau(G)$ *because* $p \nmid |F_1|$.)

Proof. Let $\chi \in Irr(G)$ and r a prime such that

$$r^{\tau(G)} \mid \chi(1).$$

If $\varphi \in Irr(F_1)$ with $\chi_{F_1} = \varphi + \cdots$ and $r^a \mid \varphi(1)$, then $a \leqslant \tau_p(F_1) \leqslant \tau_p(G)$ because $\tau_p(F_1) = \tau(F_1)$. In particular

$$r^{(\tau(G) - \tau_p(G))} \mid |G/F_1|$$

and (ii) implies

$$\tau(G) \leqslant (n(G) - 2)(4\tau_p \log_2 p + 2\tau_p) + 3\tau_p.$$

Assertion (c) follows by (a).

3.3. *Remarks.* (a) The bounds in Theorem 3.2 are obviously not sharp. It is possible to improve the bounds if we consider the group structure more precisely. An interesting question is whether it is possible to improve the bound for $\tau(G)$ such that $\tau_p(G)$ occurs only linear. The answer depends on the following question: Can be bound $\tau(G)$ without using a bound for $n(G)$ (which depends on $\tau_p(G)$ too)?

(b) The bound in Theorem 3.2(c) depends on the prime p and not only on $\tau_p(G)$. Is it possible to find a bound which is independent of p?

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